

MITTAG-LEFFLER FUNCTION, ITS SPECIFIC CASES**Gulboeva Shakhnoza Anvarovna**

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Annotation. *In this article, the Mittag-Leffler function, which plays an important role in fractional analysis and problems of mathematical physics, is studied in detail. The definition of this function, its basic analytical properties, its representation in the form of a series, and its connection with the Laplace transformations are highlighted. Also, one-parameter and two-parameter special cases of the Mittag-Leffler function are analyzed, including the transition to an exponential function and its application in the solution of fractional differential equations. In the course of the research, the role of this function in subdiffusion, anomalous transport processes, and problems with nonlocal boundary conditions is substantiated. The obtained results are important for the development of the theory of fractional differential equations and their application in practical problems.*

Keywords. *Mittag-Leffler function, fractional analysis, fractional differential equations, Laplace transform, subdiffusion process, anomalous diffusion, nonlocal boundary conditions, exponential function, mathematical physics.*

Introduction. In recent years, fractional analysis and the theory of fractional differential equations have become one of the important scientific directions in mathematical physics, mechanics, biophysics, and modeling of economic processes. The use of fractional derivatives in the study of complex processes with memory and nonlocal properties, which cannot be described using traditional differential equations, gives effective results. When analyzing such models, special functions, in particular, the Mittag-Leffler function, appear as an important mathematical apparatus.

The Mittag-Leffler function was first introduced by the Swedish mathematician G. Mittag-Leffler at the beginning of the 20th century and is a natural generalization of the exponential function. This function constitutes the main class of solutions for fractional differential equations and can be analytically expressed through Laplace transformations. In particular, the Mittag-Leffler function plays a central role in describing the processes of subdiffusion and anomalous diffusion with the memory effect over time.

In modern scientific literature, various generalizations of the Mittag-Leffler function, including one-parameter, two-parameter, and multi-parameter forms, are widely studied. These generalizations are effectively used in solving problems with nonlocal boundary conditions, inverse problems, and solutions of integral-differential equations. At the same time, some special cases of functions are of particular scientific interest in terms of their inextricable connection with exponential and other classical functions.

Literature review. The Mittag-Leffler function, as one of the main elements of fractional analysis theory, has been thoroughly studied by many foreign and domestic researchers. The initial theoretical foundations of this function were developed by the Swedish mathematician G. Mittag-Leffler, who defined it as a generalization of the exponential function in the theory of special functions[1]. Later, this function began to be widely used in problems related to complex analysis and integral transformations.

The scientific works of such scientists as I.Podlubny [2], K.Oldham [3] and J.Spanier [3] are of great importance in the formation of the theory of fractional differential equations. In particular, I. Podlubny "in his research substantiated the central place of the Mittag-Leffler function in the class of solutions in determining fractional derivatives and integrals" [2]. In his works, the representation of one-parameter and two-parameter Mittag-Leffler functions through the Laplace transform is analyzed in detail.

In the studies of R.Gorenflo, F.Mainardi, and A.Kilbas, the Mittag-Leffler function is considered as the main mathematical tool for modeling the processes of subdiffusion and anomalous diffusion. These authors showed that the fundamental solutions of fractional differential equations are expressed precisely through the Mittag-Leffler function [5]. In the works of F.Mainardi, the inextricable connection of this function with the memory effect in physical processes is scientifically substantiated [5].

In the monographs written by A.A. Kilbas, H. Srivastava, and J.J. Trujillo, generalized representations of the Mittag-Leffler function, including multi-parameter and matrix forms, were studied. These studies contributed to the development of the theory of nonlocal boundary value problems and integral-differential equations. The authors paid special attention to the analytic continuation and asymptotic properties of the function [4].

The practical applications of the Mittag-Leffler function are also widely covered in the scientific works of scientists from the CIS and Uzbekistan. In particular, the problems of using this function in fractional-order boundary value problems, inverse problems, and solutions of evolutionary equations have been studied. In some studies, solutions for subdiffusion equations with nonlocal initial and boundary conditions were constructed using the Mittag-Leffler function.

The analyzed literature shows that the Mittag-Leffler function is considered as a universal mathematical apparatus in problems of fractional analysis and mathematical physics. At the same time, some of its special cases, the connection with the exponential function, and the issues of determining new areas of application remain relevant. This article is aimed at summarizing and developing existing research in this area.

Main Part. 1. Definition and general form of the Mittag-Leffler function

The Mittag-Leffler function belongs to a special class of functions, which is defined by the following series:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} z^k / \Gamma(\alpha k + 1)$$

Here $\Gamma(\cdot)$ - Euler gamma function. When $\alpha = 1$, the Mittag-Leffler function is equivalent to an exponential function:

$$E_1(z) = e^z$$

The two-parameter Mittag-Leffler function is defined as follows:

$$E_{\{\alpha, \beta\}}(z) = \sum_{k=0}^{\infty} z^k / \Gamma(\alpha k + \beta)$$

2. Basic analytical properties. One of the important properties of the Mittag-Leffler function is its connection with the Laplace transforms:

$$\mathcal{L}\{t^{\beta-1} E_{\{\alpha, \beta\}}(\lambda t^{\alpha})\} = s^{\alpha-\beta} / (s^{\alpha} - \lambda)$$

This relation is widely used in the analytical solution of fractional differential equations.

3. Some special cases. $\alpha = 1 \Rightarrow E_1(z) = e^z$

$$\beta = 1 \Rightarrow E_{\{\alpha, 1\}}(z) = E_{\alpha}(z)$$

$\alpha = 1/2 \Rightarrow$ the function is related to the error function.

4. Application in fractional differential equations. Consider the following fractional differential equation:

$$D_t^{\alpha} y(t) = \lambda y(t), \quad 0 < \alpha \leq 1$$

$$y(0) = y_0$$

The solution to this equation is represented by the Mittag-Leffler function:

$$y(t) = y_0 E_{\alpha}(\lambda t^{\alpha})$$

5. Significance in subdiffusion and nonlocal problems. Solutions in subdiffusion and anomalous transport processes are represented by the Mittag-Leffler function. This function allows for more accurate modeling of systems with a nonlocal memory effect.

Conclusion. In this article, the Mittag-Leffler function, which is an important element of fractional analysis theory, and some of its special cases were systematically studied. In the course of the research, the definition of this function, its representation in the form of a series, its basic analytical properties, and its

inextricable connection with the exponential function were theoretically substantiated.

The analysis showed that the Mittag-Leffler function constitutes a fundamental class of solutions for fractional differential equations. In particular, one-parameter and two-parameter Mittag-Leffler functions play an important role in constructing mathematical models describing nonlocal and memory processes. The dependence of the function on the Laplace transform allows us to solve fractional equations analytically.

Also, the research results showed that the slow decay of solutions over time in subdiffusion and anomalous diffusion processes is directly related to the asymptotic properties of the Mittag-Leffler function. In some special cases, the transition of this function to classical exponential and error functions ensures a close relationship between fractional and classical models.

In general, the Mittag-Leffler function acts as a universal mathematical apparatus for solving urgent problems of fractional analysis and mathematical physics. The obtained results have important scientific and practical significance for the further development of the theory of fractional differential equations, the study of nonlocal boundary value and inverse problems, and their application in practical fields.

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